Note: * \( \beta \) is a matrix composed of \( \beta_{ij} \) \((i,j = 1, 2, 3)\)
* \( \beta_{ij} \) for \( \{i = 1, 2, 3\} \{j = 1, 2, 3\} \) denote the nine scalar components of 3x3 matrix \( \beta \)

Similarly: * \( \delta \) itself is a scalar for a given choice of \( i \) and \( j \).
* But \( \delta = \delta_{ij} \) for \( \{i = 1, 2, 3\} \{j = 1, 2, 3\} \) is the identity matrix.

Let’s note some properties of a transformation matrix \( \beta \)

\[
\begin{align*}
|\mu|^2 &= \overline{u}_m \overline{u}_m = (\beta_{mk} \mu_k) (\beta_{me} \mu_e) \\
&= u_k \mu_k = U_K (\delta_{kl} \mu_e)
\end{align*}
\]

since this is true for any vector \( \mu \), then

\[
\begin{align*}
\beta_{mk} \beta_{me} &= \delta_{ke} \\
\beta_{km} \beta_{me} &= \delta_{kl}
\end{align*}
\]

Note that: \( \alpha_{ij} \beta_{jk} \) corresponds to usual matrix multiplication

Or I can write the expression for all nine choices of \( (k,l) \) all at once using matrix notation

\[
\beta^T \beta = \delta = \text{identity matrix}
\]
Thus \( \beta^T = \beta^{-1} \)

i.e. the inverse of a transformation matrix is its transpose

\[ \implies \beta \text{ is an orthogonal matrix!} \]

Also: recall a few things from linear algebra:
(\( a, b, c \) are matrices and \( c = ab \))
then
1. \( \det(a) = \det(a^T) \)
2. \( \det(c) = \det(ab) = \det(a) \det(b) \)
3. \( \det(S) = 1 \quad s = \text{identity matrix} \)

So:
\[
\beta^T \beta = S \\
\det(\beta^T \beta) = \det(S) = 1 \\
\det(\beta^T) \det(\beta) = 1 \\
\det(\beta) \det(\beta) = 1 \\
[\det(\beta)]^2 = 1 \\
\det(\beta) = \pm 1 \]
* we define a deformable solid as a "continuum"

* define mass density at point $P$

\[
p(P) = \lim_{R \to 0} \left( \frac{\text{mass of sphere}}{\text{volume of sphere}} \right)
\]

"continuum" means we assume $p(P)$ to be a continuous function of $P$. Taken literally -- this is nonsense!

Idea is useful if we interpret $R \to 0$ to mean that $R$ becomes smaller than what we resolve or what we care about.

* Single crystal of Al, Cu, Si

\[a \sim 5 \text{Å} \quad \text{Å} = 10^{-10} \text{m} \]

nuclear radius is $\sim 10^{-4}$ atomic radius

$L \sim $ is a "continuum" provided $R \geq 10a$

* polycrystal of Al, Fe

\[d \sim 10-100 \mu\text{m} \quad 1 \mu\text{m} = 10^{-6} \text{m} \]

"continuum" provided $R \geq 10d$
so: it is necessary to consider a length scale to define a continuum!

- fiber reinforced composite material
  - fiber spacing, diameter
- muscle or bone tissue
  - size of a cell

stress in a solid

* normal vector \( \mathbf{n} \) - unit vector that specifies the orientation of a material surface at a point (usually it points "outwards")

* traction vector \( \mathbf{T}(\mathbf{n}) \) - force per unit area acting on a material surface described by \( \mathbf{n} \)

eg) a beam

\[ \mathbf{T}(\mathbf{n}) \text{ is a vector function over a surface} \]

The traction vector describes internal (force/area) that neighboring particles of a continuous material exert on the particle.
in general

\[ T(\hat{n}) = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A} \]

\[ T(\hat{n}) \text{ is a vector, whose components have units } F/L^2 \]

* at the same point in the material (if the point is stationary)

\[ T(-\hat{n}) = -T(\hat{n}) \]

* traction can be defined on an interior surface or on the exterior of a solid

* consider rectangular cartesian coordinate system in a solid. The traction components on planes defined by \((e_1, e_2, e_3)\) have a special significance

\[ T(e_1) = \sigma'_{11} e_1 + \sigma'_{12} e_2 + \sigma'_{13} e_3 \]

\[ T(e_2) = \sigma'_{21} e_1 + \sigma'_{22} e_2 + \sigma'_{23} e_3 \]

\[ T(e_3) = \sigma'_{31} e_1 + \sigma'_{32} e_2 + \sigma'_{33} e_3 \]

* definition of stress tensor
the $\sigma_{ij}$ are particular components of the traction vector

\[ \sigma \]

$\sigma_{ij}$

direction of traction

surface normal

Claim: if we know $\sigma_{ij}$ at a point for all $(i,j)$ then we can find the traction vector on any plane through that point

\[ \mathbf{T}(\mathbf{n}) = T_i(\mathbf{n}) \mathbf{e}_i \]

$\mathbf{n} = n_i \mathbf{e}_i$

area $\Delta A$

At equilibrium, the total force acting on our point is zero

\[ \Rightarrow \mathbf{T}(\mathbf{n}) \Delta A + \mathbf{T}(-e_1) \mathbf{n}_1 \Delta A + \mathbf{T}(-e_2) \mathbf{n}_2 \Delta A + \mathbf{T}(-e_3) \mathbf{n}_3 \Delta A = 0 \]