the any stress function is \( \Phi \), we'll need:

\[
\sigma'_{11} = \frac{\partial^2 \Phi}{\partial x_1^2} \\
\sigma'_{22} = \frac{\partial^2 \Phi}{\partial x_2^2} \\
\sigma'_{12} = \frac{\partial^2 \Phi}{\partial x_1 \partial x_2}
\]

and we require that

\[
\nabla^2 \Phi = \frac{\partial^4 \Phi}{\partial x_1^4} + 2 \frac{\partial^4 \Phi}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 \Phi}{\partial x_2^4} = 0
\]

\( \Phi \) is a scalar function of \( x_1, \) \( x_2 \).

consider a general form of \( \Phi \)

\[
\Phi = \frac{A}{6} x_1^3 + \frac{B}{2} x_1^2 x_2 + \frac{C}{2} x_1 x_2^2 + \frac{D}{6} x_2^3
\]

1) consider as general form does it satisfy the biharmonic equation?

\[
0 + 0 + 0 + 0 = 0 \quad \text{heyhey!}
\]

note that clearly this or any lower order polynomials would work.

2) what are the stresses generated by this function?

\[
\sigma'_{11} = \frac{\partial^2 \Phi}{\partial x_1^2} = C x_1 + D x_2
\]

\[
\sigma'_{22} = A x_1 + B x_2
\]

\[
\sigma'_{12} = -B x_1 - C x_2
\]

significance? we have now the solution for any problem with stresses at boundary that vary linearly w/ position.

eg) pure bending

\[
\sigma'_{11} = D \Delta \\
A = B = C = 0
\]

\( D \neq 0, \sigma'_{11} = D x_2 \)
\[ \sigma_{22} = Bc \]
\[ \sigma_{12} = Bx_1 \]
\[ \sigma_{12} = Bx_1 \]
\[ \sigma_{22} = -Bc \]

Main point: must find any stress function that generates stresses to match the boundary conditions.

Consider a more complicated form for \( \Phi \):
\[
\Phi = \frac{A}{20} x_1^5 + \frac{B}{12} x_1^4 x_2 + \frac{C}{6} x_1^3 x_2^2 + \frac{D}{6} x_1^2 x_2^3 + \frac{E}{12} x_1 x_2^4 + \frac{F}{20} x_2^5
\]

This will generate solutions to the harmonic equation? Yes, for appropriate choice of \( A, B, C, D, E, F \).

\[
\begin{align*}
\frac{\partial^2 \Phi}{\partial x_1^4} &= 6Ax_1 + 2Bx_2 \\
\frac{\partial^2 \Phi}{\partial x_1^2 \partial x_2^2} &= 2Cx_1 + 2Dx_2 \\
\frac{\partial^2 \Phi}{\partial x_2^4} &= 2Ex_1 + 6Fx_2
\end{align*}
\]
\[
\nabla^4 \Phi = 0
\]
\[
\begin{align*}
6Ax_1 + 2Bx_2 + 4Cx_1 + 4Dx_2 + 2Ex_1 + 6Fx_2 &= 0 \\
x_1(6A + 4C + 2E) + x_2(2B + 4D + 6F) &= 0
\end{align*}
\]

\( \Rightarrow \) require \[
E = -\left(3A + 2C\right) \\
F = -\left(\frac{1}{3}B + \frac{2}{3}D\right)
\]
to generate an equilibrium stress field.
How do you know what order polynomial to choose for any stress function? From inspection of boundary conditions.

\[ \text{3rd order polynomial } \varphi \text{ gives linearly varying stresses} \]

\[ \sigma_{12} \sim x_1^3 \]

need at least 5th order polynomial to get cubic varying b.c.'s.

Example: end-loaded cantilever

\[
\begin{align*}
\text{L}^2 \text{plane stress } & \quad \sigma_{33} = 0, \quad \sigma_{11}(x_1, x_2), \quad \sigma_{12}(x_1, x_2), \quad \sigma_{22}(x_1, x_2) \\
\text{L}^P \text{ use } & \quad \varphi = -\frac{3P}{4a b} x_1 x_2, \quad + \frac{P}{4a^3 b} x_1 x_2 \\
\text{does it satisfy } & \quad \nabla^4 \varphi = 0? \\
& \quad \frac{\partial^4 \varphi}{\partial x_1^4} + \frac{\partial^4 \varphi}{\partial x_2^4} + \frac{2\partial^4 \varphi}{\partial x_1^2 \partial x_2^2} = 0 \quad \checkmark
\end{align*}
\]

\[ \text{find the stresses} \]

\[ \sigma_{11} = \frac{\partial^2 \varphi}{\partial x_2^2} = \frac{3P}{2a^3 b} x_1 x_2 \]

\[ \sigma_{22} = \frac{\partial^2 \varphi}{\partial x_1^2} = 0 \]

\[ \sigma'_{12} = \sigma'_{21} = -\frac{\partial^2 \varphi}{\partial x_1 \partial x_2} = \frac{3P}{4ab} \left(1 - \frac{x_2^2}{a^2}\right) \]
can we satisfy the bc's with this stress function?

1) traction-free at \( x_2 = \pm a \) \((\sigma_{ij} n_j = 0, \pm \epsilon_{x_2} \text{ normal})\)
\[
\sigma_{z2} = \sigma_{21} = 0 \quad \checkmark
\]
\( x_2 = \pm a \)

2) plane stress assumption satisfies traction free boundaries on \( x_3 = \pm b/2 \)

3) traction boundary condition on left end \((x_1 = 0)\)
\[
t_1 = \sigma_{ij} n_j \Rightarrow t_1 = \sigma_{21} n_1 + \sigma_{22} n_2
\]
\( \sigma_{ij} = 0 \text{ at } x_1 = 0 \)
\( n_2 = 0 \)
\( t_1 = 0 \quad \checkmark \)
\[
t_2 = \sigma_{21} n_1 + \sigma_{22} n_2 = -\sigma_{21}
\]
so then \( T \)
\[
= -\frac{3P}{4ab} \left(1 - \frac{x_2^2}{a^2}\right) \epsilon_{x_2}
\]
and \( F \)
\[
= b \int_{-a}^{a} \left(-\frac{3P}{4ab} \left(1 - \frac{x_2^2}{a^2}\right)\right) dx_2 = -P \quad \checkmark
\]

* but note that for \( \Phi \) to be exactly the correct solution, \( P \) not applied at a point as illustrated above, but should be distributed according to \( T \) on left face

4) traction boundary condition on right-face? not specified, but settle for
\[
u_1 = 0 \quad 2u_2/\partial x_1 = 0
\]
\( u_2 = 0 \)
- on right side, the boundary condition is a displacement condition: 
  \( u_1 = 0, u_2 = 0, \) 
  \( \partial u_2 / \partial x_1 = 0 \)

From the Airy Stress Function, the exact solution is

\[
\begin{align*}
  u_1 &= \frac{3P}{4Ea^3b} x_1^2 x_2 - \frac{P}{4Ea^3b} (2+\nu) x_2^3 \\
  &\quad + \frac{3P}{2Ea^3b} (1+\nu) a^2 x_2 - w x_2 + C \\
  u_2 &= -v \frac{3P}{4Ea^3b} x_1 x_2^2 - \frac{P}{4Ea^3b} x_1^3 + w x_1 + d
\end{align*}
\]

Note: \( C, d, w \) are constants.