Fundamental Concepts in Solid Mechanics

Material properties → stress → equilibrium → BVPs → displacement → strain → compatibility

to provide a physical context: materials

The tension test

\[ \sigma = \frac{P}{A} \quad \text{strain} \quad e = \frac{\Delta l}{l} \]

If \( A = 1 \text{ cm}^2 = 10^{-4} \text{ m}^2 \) and \( l = 1 \text{ m} \): at yield \( S = 1 \text{ mm} \)

\( P = 20,000 \text{ N} \)
we will use the mathematical language and notation of continuum mechanics

**Scalars** ~ temperature, arc length, elastic modulus, ...

**Vectors** ~ velocity, force, electric field, normal

e.g. \( \mathbf{v} \) vector in 3D space

\[ |\mathbf{v}| = \| \mathbf{v} \| = \text{magnitude of } \mathbf{v} \]

if \( |\mathbf{v}| = 1 \), \( \mathbf{v} \) is a **unit vector**

how do we calculate w/ vectors?
- introduce a coordinate system
- introduce a basis (ours will be orthonormal)

eg) Cartesian coordinate system
   - basis vectors are unit vectors along the three \( x \) coord. directions

**ORTHO-NORMAL** means

\[ \mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} \]

where \( i, j = 1, 2, 3 \) and

\[ \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases} \]

"Kronecker Delta"
Any vector can be expressed:

\[ \vec{u} = u_1 \vec{e}_1 + u_2 \vec{e}_2 + u_3 \vec{e}_3 \]

Scalar components of \( \vec{u} \)

\[ \sum_{k=1}^{3} u_k \vec{e}_k = \sum_{m=1}^{3} u_m \vec{e}_m = u_k \vec{e}_k \]

We will use **Einstein summation convention**

A repeated index in an expression implies that the expression is summed over that index.

Thus, let's review some vector operations using Einstein notation

**Inner Product / Dot Product**

\[ \vec{u} \cdot \vec{v} = \left( \sum_{k=1}^{3} u_k \vec{e}_k \right) \cdot \left( \sum_{m=1}^{3} v_m \vec{e}_m \right) \]

\[ = \sum_{k=1}^{3} u_k v_m \vec{e}_k \cdot \vec{e}_m \]

\[ = \sum_{k=1}^{3} u_k v_m \delta_{km} \]

\[ = u_k v_k \quad \ldots \quad = u_1 v_1 + u_2 v_2 + u_3 v_3 \]
\[ |\mathbf{u}| = \sqrt{\mathbf{u} \cdot \mathbf{u}} = \sqrt{\mathbf{u}_p \cdot \mathbf{u}_p} = \sqrt{u_1^2 + u_2^2 + u_3^2} \]

**Vector Product / Cross Product**

\[ \mathbf{e}_i \times \mathbf{e}_j = \varepsilon_{ijk} \mathbf{e}_k \]

where \( \varepsilon_{ijk} = \begin{cases} +1 & \text{if } ijk = 123, 231, 312 \\ -1 & \text{if } ijk = 132, 213, 321 \\ 0 & \text{otherwise} \end{cases} \)

\( \varepsilon \) is called the "permutation operator"

Check:

\[ \mathbf{e}_1 \times \mathbf{e}_2 = \varepsilon_{123} \mathbf{e}_3 = \mathbf{e}_3 \]

\[ \mathbf{e}_3 \times \mathbf{e}_2 = \varepsilon_{321} \mathbf{e}_1 = -\mathbf{e}_1 \]

\[ \mathbf{e}_2 \times \mathbf{e}_2 = 0 \]

Then

\[ \mathbf{u} \times \mathbf{v} = (u_i \mathbf{e}_i) \times (v_j \mathbf{e}_j) = u_i v_j (\mathbf{e}_i \times \mathbf{e}_j) = u_i v_j \varepsilon_{ijk} \mathbf{e}_k \]

**Gradient of a Scalar Field** \( f(x_1, x_2, x_3) \)

\[ \nabla f = \frac{\partial f}{\partial x_1} \mathbf{e}_1 + \frac{\partial f}{\partial x_2} \mathbf{e}_2 + \frac{\partial f}{\partial x_3} \mathbf{e}_3 \]

\[ = \frac{\partial f}{\partial x_k} \mathbf{e}_k \]
In general the gradient operator does
\[ \nabla = \frac{\partial}{\partial x_i} e_i + \ldots = \frac{\partial}{\partial x_k} e_k \]

\( \nabla \cdot \mathbf{u} = \frac{\partial u_i}{\partial x_i} \)

\( \nabla \times \mathbf{u} = \varepsilon_{ijk} \frac{\partial u_j}{\partial x_k} e_i = \partial_i u_j e_k e_{ijk} = u_{j,i} e_k e_{ijk} \)
\[ = (\frac{\partial u_i}{\partial x_2} - \frac{\partial u_2}{\partial x_3}) e_1 + (\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}) e_2 + (\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}) e_3 \]

\underline{Coordinate Transformations}

\( \mathbf{u} \) is independent of choice of coordinates, but \( u_k \) are dependent on choice of coordinates.

Suppose a different coordinate system \( \bar{x}_k \) is selected to represent components of \( \mathbf{u} \), say \( \bar{u}_k \).

How are \( u_k (x_1, x_2, x_3) \) and \( \bar{u}_k (\bar{x}_1, \bar{x}_2, \bar{x}_3) \) related?
\[
\begin{align*}
\mathbf{P} &= P_1 \mathbf{e}_1 + P_2 \mathbf{e}_2 = P_K \mathbf{e}_K \\
\mathbf{\tilde{P}} &= \tilde{P}_1 \mathbf{\tilde{e}}_1 + \tilde{P}_2 \mathbf{\tilde{e}}_2 = \mathbf{\tilde{P}}_m \mathbf{\tilde{e}}_m
\end{align*}
\]

\(\mathbf{\tilde{c}}\) is the vector from one origin to the other.

\[
\mathbf{P} = \mathbf{P} - \mathbf{\tilde{c}}
\]

Thus

\[
\mathbf{\tilde{P}} = \mathbf{P}_m \mathbf{\tilde{e}}_m = P_K \mathbf{\tilde{e}}_K - \mathbf{\tilde{c}}
\]

dot both sides by \(\mathbf{\tilde{e}}_l\):

\[
\mathbf{\tilde{P}}_m \mathbf{\tilde{e}}_m \cdot \mathbf{\tilde{e}}_l = P_K \mathbf{\tilde{e}}_K \cdot \mathbf{\tilde{e}}_l - \mathbf{\tilde{c}} \cdot \mathbf{\tilde{e}}_l
\]

\[
\mathbf{\tilde{P}}_l = P_K \mathbf{\tilde{e}}_K - \mathbf{\tilde{c}}_l
\]

Call this \(\mathbf{\beta}_{kl}\)

Finally:

\[
\mathbf{\tilde{P}}_l = \mathbf{\beta}_{kl} P_K - \mathbf{\tilde{c}}_l
\]

How coordinates transform!

\(\mathbf{\beta}\) is called a transformation matrix.

\textbf{Note:} Scalars are invariant, i.e.

\[
\Phi(X_K) = \Phi(\mathbf{\tilde{X}}_K)
\]

\textbf{Note:} Vector components transform like:

\[
\mathbf{\tilde{u}} = u_K \mathbf{\tilde{e}}_K = \mathbf{\tilde{u}}_K \mathbf{\tilde{e}}_K
\]

\[
\mathbf{\tilde{u}}_K \mathbf{\tilde{e}}_K \cdot \mathbf{\tilde{e}}_m = \mathbf{\tilde{u}}_m
\]

\[
\mathbf{\tilde{u}}_m = \mathbf{\beta}_{mk} u_K
\]

How vector components transform!