Last time we considered a specific geometry and derived the Griffith criterion:
\[ \sigma \sqrt{\pi a} = \sqrt{2 \pi E} \equiv K_{IC} \]

Sometimes called "fracture toughness."

Now, how do we generalize this to other realistic geometries?

ie) *Introduce the stress intensity factor \( K \), which accounts for amplification/suppression of stresses due to the particular geometry:

\[ K = Y \sigma \sqrt{\pi a} \]

\( Y \) is the geometrical factor, dimensionless.

and (for mode I) if \( K > K_{IC} \) \( \Rightarrow \) unstable.

For special geometries, \( K \) has been determined, i.e.:

<table>
<thead>
<tr>
<th>GEOMETRY</th>
<th>( K_I ) (for mode I)</th>
</tr>
</thead>
<tbody>
<tr>
<td>center crack of length 2a in an infinite plate</td>
<td>( 1.12 \sigma \sqrt{\pi a} )</td>
</tr>
<tr>
<td>edge crack, length a, in a semi infinite plate</td>
<td>( 1 \sigma \sqrt{\pi a} )</td>
</tr>
<tr>
<td>central circular crack, radius a, in an infinite body</td>
<td>( 2 \sigma \sqrt{\frac{a}{\pi}} )</td>
</tr>
<tr>
<td>center crack length 2a in plate of width W</td>
<td>( \sigma \sqrt{W \tan \left( \frac{Ta}{W} \right)} )</td>
</tr>
<tr>
<td>2 symmetric edge cracks, length a,</td>
<td>( \sigma \sqrt{W \tan \left( \frac{Ta}{W} \right)} + 0.1 \sigma )</td>
</tr>
</tbody>
</table>
The idea of the stress intensity factor $K$ was introduced by Irwin, who showed that in general for any geometry we can write:

\[
\sigma_{ij} \approx \frac{K}{\sqrt{2\pi r}} f_{ij}(\theta)
\]

where:

1. $K$ = stress intensity factor, different for varying geometries and modes ($K_I, K_{II}, K_{III}$)
2. The $(2\pi r)^{-1/2}$ is universal amongst all geometries and shows the singular nature of the stress distribution
3. The angular dependence $f_{ij}$ is separable

Where does $K$-field theory apply?
* Not very close to crack tip (within a few atomic radii)
* Not very far from crack tip (near boundaries of object)

How to determine $K_I, K_{II}, K_{III}$:
1. Solve full linear elastic BVP, extract the stress intensity factor from asymptotic behavior of fields
2. Numerical models
3. Energy methods or path independent integrals
4. Look up in tables
Let's do it for mode I, plane stress or plane strain:

\[
\begin{align*}
1. & \quad \frac{\partial \sigma_r}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\sigma_{r\theta} - \sigma_{ee}}{r} = 0 \\
2. & \quad \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{2 \sigma_{r\theta}}{r} = 0
\end{align*}
\]

**EQUILIBRIUM EQUATIONS**

**COMPATIBILITY**

\[
\left( \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \left( \sigma_r + \sigma_{r\theta} \right) = 0
\]

Saw this when we derived Airy Stress Function.

**Boundary Conditions**

\[
\begin{align*}
\sigma_{ee} \left( r_1 \pm \Pi \right) &= 0 \\
\sigma_{r\theta} \left( r_1 \pm \Pi \right) &= 0
\end{align*}
\]

Since the equation is "equidimensional" we assume a power-law solution

\[
\begin{align*}
\sigma_r \left( r, \theta \right) &= r^\lambda f_{rr} \left( \theta \right) & f_{rr} \left( \theta \right) &= f_{rr} \left( -\theta \right) \\
\sigma_{r\theta} \left( r, \theta \right) &= r^\lambda f_{r\theta} \left( \theta \right) & f_{r\theta} \left( \theta \right) &= f_{r\theta} \left( -\theta \right) \\
\sigma_{ee} \left( r, \theta \right) &= r^\lambda f_{ee} \left( \theta \right) & f_{ee} \left( \theta \right) &= f_{ee} \left( -\theta \right) \\
\sigma_{r\theta} \left( r, \theta \right) &= r^\lambda f_{r\theta} \left( \theta \right) & f_{r\theta} \left( \theta \right) &= -f_{r\theta} \left( \theta \right)
\end{align*}
\]

Substitute into equations 1, 2, 3

1. \( \lambda f_{rr} + f_{r\theta}' + f_{rr} - f_{ee} = 0 \)
2. \( \lambda f_{r\theta} + f_{ee}' + 2f_{r\theta} = 0 \)
3. \( \left[ \lambda (2\lambda - 1) + 2 \right] \left( f_{rr} + f_{ee} \right) + f_{rr}'' + f_{ee}'' = 0 \)

b.c.'s: \( f_{ee} \left( \pi \right) = 0 \), \( f_{r\theta} \left( \pi \right) = 0 \)
then \( 3 \) solution is \( f_{rr} + f_{\theta\theta} = \kappa R \cos (\lambda \theta) \)

\( 2 \) \( f_{\theta\theta} = -\frac{1}{r+2} f_{\theta\theta} \)

Use these to eliminate \( f_{rr}, f_{\theta\theta} \) from \( 1 \)

\((r+1)A \cos (\lambda \theta) - (r+1) f_{\theta\theta} - \frac{1}{(r+2)} f_{\theta\theta} - f_{\theta\theta} = 0 \)

Thus: \( f_{\theta\theta} + (r+2)^2 f_{\theta\theta} = (r+1)(r+2) A \cos (\lambda \theta) \)

\[ f_{\theta\theta} = C \cos (r+2) \theta + \frac{1}{r} (r+2) A \cos (\lambda \theta) \]

\( \text{homogeneous} \) \( \text{particular} \)

Boundary condition: \( f_{\theta\theta} (\pi) = 0 \)

\( \Rightarrow C \cos (\lambda \pi) + \frac{1}{r} (r+2) A \cos (\lambda \pi) = 0 \)

\( C + \frac{1}{r} (r+2) A = 0 \)

\( \text{OR} \quad \lambda = -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2} \ldots \)

But we require the strain energy to be integrable, so

\[ \lim_{r \to 0^+} \tau^2 \sigma_{ij} \sigma_{ij} = 0 \quad \text{finite} \quad \Rightarrow \lambda > -1 \]

Thus the dominant term in the crack tip field is \( r^{-1/2} \)

(strongest singularity allowed)

Also going back to \( 1, 2, 3 \) we can show

\[ f_{\theta\theta} (\theta) = C \sin (r+2) \theta + \frac{1}{r} A \lambda \sin (\lambda \theta) \]

\[ f_{\theta\theta} (\pi) = 0 \quad \Rightarrow C + \frac{1}{r} (r+2) A = 0 \quad \text{OR} \quad \lambda = -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \ldots \]