Remember HW 5 due W Nov 29
Mid-term 2 on W Nov 29 in class
No class on Mon. Nov 27

last week: how to determine if a material will yield under a particular load

this week: what happens to a material at yield?
what is the relationship between $\sigma$ and $\varepsilon_P$?

$\dot{\varepsilon}_P =$ strain "rate"

$d\varepsilon_P =$ plastic strain "increment"

replace with

in general $d\varepsilon = d\varepsilon^e + d\varepsilon^p$
To understand what happens to a material at yield, we need two key elements:

1) a plastic flow rule
2) a strain hardening model

I. START WITH FLOW RULE

Decompose total strain rate into elastic/plastic part:

\[ \varepsilon_{ij} = \varepsilon_{ij}^e + \varepsilon_{ij}^p \]

- Reversible
- Not recoverable

The total plastic strain in the crystal is the sum of all prior increments:

\[ \varepsilon^p = \int \varepsilon^p = \int \frac{1}{3} \varepsilon^p_{ij} d\varepsilon^p_{ij} \]

Ex: Uniaxial tension

When plastic strain doesn't change volume:

\[ \varepsilon^p_{22} = \varepsilon^p_{33} = \frac{\varepsilon^p_{11}}{2} \]

The **PLASTIC FLOW RULE** relates the individual components of the plastic strain increment tensor to the total plastic strain increment:

\[ \varepsilon_{ij}^p = \varepsilon^p \frac{\partial f}{\partial \varepsilon_{ij}} \]

... will see where this comes from today.
imagine plastic flow during shear of a crystal

\[ S = \text{dislocation source} \]

The current stress determines the current direction of dislocation movement, and thus the current strain increment \( \frac{\Delta e_{ij}^p}{\Delta t} \) \( \rightarrow \)
AND NOT IT’S TOTAL VALUE \( \frac{\Delta e_{ij}}{\Delta t} \)

note a key difference wrt elastic response!!

Consider, e.g., a crystal under biaxial stress \( \sigma_{11}, \sigma_{22} \) of slip planes at 45°

\[ \sigma_{11} \quad \sigma_{22} \]

- maximum shear stress \( \tau \) occurs at 45° planes as well
- \( \tau = \frac{1}{2} | \sigma_{11} - \sigma_{22} | \)
- if \( \tau \) exceeds \( \tau_c \) then yield.
- as the crystal slips, in what direction is the strain increment?

1. \( \Delta e_{ij}^p \)

\[ \sigma_{22} = \sigma_{11} + 2\tau \]

\[ \sigma_{22} = \sigma_{11} - 2\tau \]

1. say we are on yield surface at \( \sigma_{11} = 0, \sigma_{22} > 0 \)

\[ \square \Rightarrow \square \leftarrow \Delta e_{11} < 0 \quad \Delta e_{22} > 0 \]

\[ \Delta e_{11} = -\Delta e_{22} \]

2. on yield surface at \( \sigma_{11} > 0, \sigma_{22} = 0 \)

\[ \square \Rightarrow \square \leftarrow \Delta e_{11} > 0 \quad \Delta e_{22} > 0 \]

\[ \Delta e_{11} = -\Delta e_{22} \]
Note that the direction of plastic strain increment is normal to the yield surface. It turns out that (by geometry) this is always true for any yield surface.

\( \sigma_{33} = 0 \)

\[
f = \frac{1}{2} \left[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right] - \sigma_y^2 = \frac{1}{2} \left[ 2\sigma_1^2 + 2\sigma_2^2 - 2\sigma_1\sigma_2 \right] - \sigma_y^2
\]

The condition of normality has as a consequence the plastic flow rule (which is effectively the stress-strain relationship during yield)

the proportionality between the increments of strain and the outward normal to \( f = 0 \)

\[
\frac{d\epsilon_{11}^p}{df/\sigma_{11}} = \frac{d\epsilon_{22}^p}{df/\sigma_{22}} = \text{constant} \quad d\lambda
\]

the constant of proportionality is the total plastic strain increment \( d\epsilon_P \)
Thus we get the result

\[ d\varepsilon_{ij} = d\bar{\varepsilon}_P \frac{\partial f}{\partial \varepsilon_{ij}} \]

associated plastic flow rule (largely geometric result)

Note: to make this general rule more useful we need to specify a form for \( f \) (Aris, Tresca, ...) and evaluate \( \frac{\partial f}{\partial \varepsilon_{ij}} \) directly. More later.

II. STRAIN-HARDENING MODEL

need to account for hardening - the fact that the yield stress changes as the material yields - in our theory of plasticity

in real materials, we take a phenomenological approach to describe

upon reloading the material appears to be stronger (harder)

a model for hardening should relate the size & shape of the "evolving" yield surface to the total plastic strain in the material.

two common models: ① isotropic ② kinematic
1. **isotropic**

- Upon yield, yield surface increases in size but keeps the same shape.

\[ \sigma_1, \sigma_2, \sigma_3 \]

\[ \sigma_1 \rightarrow \sigma_2 \rightarrow \sigma_3 \]

Or

\[ \sigma_1, \sigma_2, \sigma_3 \]

The yield stress is written as \( Y = Y(\bar{\varepsilon}^P) \), is a function of the amount of plastic strain in the material.

\[ \sigma_y = Y(\bar{\varepsilon}^P) \]

where \( \bar{\varepsilon}^P = \int \sqrt{\frac{2}{3}} d\varepsilon_{ij}^p d\varepsilon_{ij}^p \)

**isotropic hardening**

In practice, for \( Y(\bar{\varepsilon}^P) \) we often assume simple relationships such as:

- No hardening: \( Y = Y_0 \)
- Linear hardening: \( Y = Y_0 + h \bar{\varepsilon}^P \)
- Power law hardening: \( Y = Y_0 + h (\bar{\varepsilon}^P)^m \)

2. **kinematic**

- Upon hardening, surface translates w/o changing shape.

Rationale: Isotropic hardening doesn't account for hysteresis effects in cyclic loading (esp. Baushinger effect)