Here is a common model based on von Mises criterion:

- Recall that we can write the full stress tensor as the sum of a hydrostatic and a deviatoric part.
- The hydrostatic stress is defined as $\bar{\sigma} = \frac{1}{3} \sigma_{kk} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3}$.

\[
\begin{bmatrix}
\sigma_1 & 0 & 0 \\
0 & \sigma_2 & 0 \\
0 & 0 & \sigma_3
\end{bmatrix} =
\begin{bmatrix}
\bar{\sigma} & 0 & 0 \\
0 & \bar{\sigma} & 0 \\
0 & 0 & \bar{\sigma}
\end{bmatrix} +
\begin{bmatrix}
\sigma_1 - \bar{\sigma} & 0 & 0 \\
0 & \sigma_2 - \bar{\sigma} & 0 \\
0 & 0 & \sigma_3 - \bar{\sigma}
\end{bmatrix}
\]

$\sigma_0$ = general stress tensor in principal coords
\[\downarrow\] hydrostatic or dilational
\[\downarrow\] shear contribution, or deviatoric "J".

- Also recall the invariants of the tensor. For $J$:
  \[
  J_1 = tr(\mathbf{J}) = (\sigma_1 - \bar{\sigma}) + (\sigma_2 - \bar{\sigma}) + (\sigma_3 - \bar{\sigma})
  \]
  \[
  J_2 = (\sigma_1 - \bar{\sigma})(\sigma_2 - \bar{\sigma}) + (\sigma_2 - \bar{\sigma})(\sigma_3 - \bar{\sigma}) + (\sigma_3 - \bar{\sigma})(\sigma_1 - \bar{\sigma})
  \]
  \[
  J_3 = \det J = (\sigma_1 - \bar{\sigma})(\sigma_2 - \bar{\sigma})(\sigma_3 - \bar{\sigma})
  \]

**Yield Will Occur**

- Now consider the strain energy density associated with the stress state $\sigma_0$

\[
U_0 = \frac{1}{2} \sigma_{ij} \epsilon_{ij} = \frac{1}{2} \sigma_1 \epsilon_1 + \frac{1}{2} \sigma_2 \epsilon_2 + \frac{1}{2} \sigma_3 \epsilon_3
\]

\[
\epsilon_{ij} = \frac{1}{2} \left( \sigma_{ij} + \sigma_{ji} - \frac{\sigma_{kk}}{3} S_{jk} \right)
\]

Recall

\[
\epsilon_{ij} = \frac{1}{2} \left( \sigma_{ij} + \sigma_{ji} - \frac{\sigma_{kk}}{3} S_{jk} \right) = \frac{1}{2E} (\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - \frac{\nu}{E} (\sigma_1 \sigma_2 + \sigma_2 \sigma_3 + \sigma_3 \sigma_1)
\]

- The strain energy density associated with only the hydrostatic part is $U_{\text{dilatational}} = \frac{3(1-\nu)}{2E} \bar{\sigma}^2 = \frac{1-2\nu}{6E} (\sigma_1^2 + \sigma_2^2 + \sigma_3^2)$

(substitute in $\sigma_1 = \sigma_2 = \sigma_3 = \bar{\sigma}$ in previous expression)
we will call the difference between the actual strain energy density and the strain energy density due to the hydrostatic part as

\[ U_{\text{deviatoric}} = U_0 - U_{\text{dilational}} \]

\[ = \frac{1+\nu}{6E} \left[ (\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2 \right] \]

(interesting to note that this is \( J_2 \))

Mises assumption: yield occurs when \( U_{\text{deviatoric}} \) reaches some critical value

What is that critical value? We can find it from the uniaxial tension experiment.

\[ \sigma = \begin{bmatrix} \sigma_1 & \sigma_2 & \sigma_3 \\ \sigma_2 & \sigma_1 & \sigma_3 \\ \sigma_3 & \sigma_2 & \sigma_1 \end{bmatrix} \]

ie) \( \sigma_1 = \sigma_2 = 0 \)

\( \sigma_3 = \sigma_y \)

\[ U_{\text{critical}} = \frac{1+\nu}{6E} \left[ \sigma_y^2 + \sigma_y^2 \right] = \frac{1+\nu}{3E} \sigma_y^2 \]

\[ U_{\text{deviatoric}} = \frac{1+\nu}{3E} \left( 2\tau_{\text{cr}} \right)^2 \]

since \( \tau_{\text{cr}} = \frac{1}{2}(\sigma_y - \sigma) \)

\[ = \frac{1+\nu}{3E} \left( 2\tau_{\text{cr}} \right)^2 \]

So yield occurs when \( U_{\text{deviatoric}} = U_{\text{critical}} \)

\[ (\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2 = 2\sigma_y^2 \]
or, when written as a yield function

\[ f_{\text{Mises}}(\sigma_1', \sigma_2', \sigma_3') = \frac{1}{2} \left[ (\sigma_1' - \sigma_2')^2 + (\sigma_2' - \sigma_3')^2 + (\sigma_3' - \sigma_1')^2 \right] - \sigma_y^2 \]

In a 3D stress space of \( \sigma_1', \sigma_2', \sigma_3' \) coordinates, \( f = 0 \) is a cylinder with axis along \( \sigma_1' = \sigma_2' = \sigma_3' \) (hydrostatic axis).

Mises yield surface is a cylinder with radius \( \sqrt{\frac{2}{3}} \sigma_y \) around hydrostatic axis.

This is the most common failure theory used in industry! A model derived back in 1913 (for ductile systems).

Some applications of von Mises theory to common states of stress:

1. Pure shear \( \tau_{12} = \tau_{21} = \tau \) (all other \( \sigma_{ij} = 0 \))

   * Find principal stresses

   \[
   \begin{vmatrix}
   -\tau & \tau & 0 \\
   \tau & -\tau & 0 \\
   0 & 0 & -\tau
   \end{vmatrix} = 0 \quad \Rightarrow \quad -\tau^2 + 2\tau^2 = 0
   \]

   \( \tau = \pm \tau \)

   i.e., \( \sigma_1 = \tau, \sigma_2 = 0, \sigma_3 = -\tau \)

   * Evaluate Mises

   \[
   f = \frac{1}{2} \left[ 4\tau^2 + \tau^2 + \tau^2 \right] - \sigma_y^2 = 0
   \]

   \( \Rightarrow \) yield when \( \tau = \pm \sigma_y / \sqrt{3} \)
2. Combined tension & shear

![Diagram of tension and shear forces]

- Find principal stresses

\[
\begin{vmatrix}
\sigma_x & \tau & 0 \\
\tau & \sigma_y & 0 \\
0 & 0 & \sigma_z \\
\end{vmatrix} = 0
\]

\[
\Rightarrow \sigma_1, \sigma_2, \sigma_3 = 0, \frac{1}{2} \sigma ± \sqrt{\sigma^2 + 4\tau^2}
\]

- Evaluate Mises criterion for shear

\[
f_{\text{Mises}} = \sigma^2 + 3\tau^2 = \sigma_y^2
\]

(with \( \sigma \) present, \( \tau \) for yield is now reduced)

3. Biaxial tension

![Diagram of biaxial tension forces]

- Principal stresses: \( \sigma_1, \sigma_2, 0 \)

\[
f = \frac{1}{2} \left[ (\sigma_1 - \sigma_2)^2 + \sigma_1^2 + \sigma_2^2 \right] - \sigma_y^2 = 0
\]

\[
1 = \left( \frac{\sigma_1}{\sigma_y} \right)^2 - \left( \frac{\sigma_1}{\sigma_y} \right) \left( \frac{\sigma_2}{\sigma_y} \right) + \left( \frac{\sigma_2}{\sigma_y} \right)^2
\]

Equation of an ellipse

![Diagram of ellipse equation]

\( \sigma_1, \sigma_y \) inaccessible.
last time: von Mises criterion for onset of plasticity

* plastic deformation when the part of the strain energy density due to deviatoric stress exceeds the critical value.

* critical value determined from uniaxial tension response and yield stress $\sigma_y$

$$f = \frac{1}{2} \left[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right] - \sigma_0^2 = 0$$

* also called **Maximum Distortion Energy Theory**

Now: an alternative yield function: **Tresca or Maximum Shear Stress Theory**

$\Rightarrow$ yield when a critical value of shear stress is reached

what is that critical value?

\[ \n_{\text{crit}} = \left| \frac{1}{2} (\sigma_y - 0) \right| = \frac{1}{2} \sigma_y \]

example: biaxial stress

\[ \sigma_2 \]

\[ \sigma_3 \]

\[ \sigma_1 \]

\[ \sigma \]

\[ \text{case 1: } \text{sign of principal stresses are the same} \]

\[ \sigma_{\text{max}} = \frac{1}{2} \sigma' \]

so \[ \frac{1}{2} \sigma_y > \frac{1}{2} \sigma' \] for no yielding

$\Rightarrow$ \[ 2 n_{\text{crit}} > \sigma' \]
Case 2: signs of principal stresses different $\sigma_1 > 0$ $\sigma_2 < 0$

$$\gamma_{\text{max}} = \frac{1}{2} (\sigma_1 - \sigma_2)$$

to avoid yield $\gamma_{\text{max}} < \gamma_{\text{cr}}$

$$\frac{1}{2} (\sigma_1 - \sigma_2) \leq \gamma_{\text{cr}} = \frac{1}{2} \sigma_y$$

Together, the two cases give a "yield surface" for biaxial loading.

In general, Tresca:

$$f = \max \left[ \frac{1}{2} |\sigma_1 - \sigma_2|, \frac{1}{2} |\sigma_1 - \sigma_3|, \frac{1}{2} |\sigma_2 - \sigma_3| \right] - \frac{1}{2} \sigma_y$$

Adding hydrostatic stress doesn't change anything. (Mohr's circle just shifts along axis.)